High velocity frame transformations
III: Properties of frame transformations

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Abstract: *The projector operator formalism is used to deduce the relationships between all the quantities involved in the high velocity frame transformations obeying suitable kinematical and symmetry conditions. A characterization of Marinov transformation is pointed out in terms of the spatial transformation matrix and its inverse.*

1. Introduction

The first two papers of this series [1,2], from now on denoted by I and II respectively, develop a convenient projector operator formalism which allows us to reformulate the kinematical and symmetry conditions used by Mansouri and Sexl [3] for simplifying the frame transformation between a privileged observer and a generic inertial one. In so doing we arrived at the final expression for the frame transformation between inertial observers, which depends on three arbitrary functions.

To further analyze the resulting theory, it is convenient to have at our disposal a collection of formulas that link the quantities involved in the four dimensional transformation and its inverse. As a first step we write down, in full generality, all the relations between such quantities without any “a priori” restrictions. Then we specialized all calculations to the case when the spatial transformation matrix takes on the form of Mansoury and Sexl, that we deduced in II. In this case we are able to determine the explicit form of the inverse four dimensional transformation as well as the inverse matrix of the spatial part. With the aid of these formulas, it is possible to deduce a peculiar characterization of Marinov transformation [4]

In the sequel we freely use notations and results proved in I and II.

2. The frame transformation

Consider two inertial frames: the absolute inertial frame A and a generic inertial frame F. Denote by x^α and X^α the space-time coordinates of A and F, respectively. In the following Greek indices run from 0 to 3, Latin indices from 1 to 3 and $x^0 = ct$, $X^0 = cT$; in principle the constant c need not be the light speed in vacuum, it suffices that c is an universal constant speed. Let v^i be the velocity of F with respect to A, namely the velocity measured by A of the points at rest in F. Analogously we denote by V^i the velocity of A with respect to F.

In ref [5] we proved that the most general transformation between two inertial frames is linear; hence we can write

(2.1) $$X^{\alpha} = \Lambda^{\alpha}{}_{\beta} x^{\beta}$$

On appealing to the velocity addition theorem we proved that [5]

(2.2) $$\Lambda^{\alpha}{}_{\beta} = \begin{pmatrix} N & cH_j \\ NV^i / c & S^i{}_j \end{pmatrix}$$

Therefore the transformation (2.1) can be split as

(2.3a) $$X^i = S^i{}_j x^j + NV^i t$$

(2.3b) $$T = H_k x^k + Nt$$

Consider now the inverse transformation

(2.4) $$x^{\alpha} = \lambda^{\alpha}{}_{\beta} X^{\beta}$$

and set

(2.5) $$\lambda^{\alpha}{}_{\beta} = \begin{pmatrix} n & ch_j \\ nv^i / c & s^i{}_j \end{pmatrix}$$

Accordingly the transformation (2.4) can be split as

(2.6a) $$x^i = s^i{}_j X^j + nv^i T$$

(2.6b) $$t = h_k X^k + nT$$

We note that this kind of notation allows us to adopt a duality rule that in any given equation majuscules and minuscules may be interchanged. Our goal is now that of writing down explicitly the conditions that $\Lambda^{\alpha}{}_{\beta}$ and $\lambda^{\alpha}{}_{\beta}$ are the inverse matrices of each other, namely

(2.7) $$\Lambda^{\alpha}{}_{\mu} \lambda^{\mu}{}_{\beta} = \delta^{\alpha}{}_{\beta} \quad \text{and} \quad \lambda^{\alpha}{}_{\mu} \Lambda^{\mu}{}_{\beta} = \delta^{\alpha}{}_{\beta}$$

Starting from the first condition (2.7), a straightforward calculation yields the following relations

(2.8a) $$\Lambda^0{}_{\mu} \lambda^{\mu}{}_0 = 1 \quad \Rightarrow \quad Nn + nH_k v^k = 1$$

(2.8b) $$\Lambda^i{}_{\mu} \lambda^{\mu}{}_0 = 0 \quad \Rightarrow \quad NV^i + S^i{}_k v^k = 0$$

(2.8c) $$\Lambda^0{}_{\mu} \lambda^{\mu}{}_j = 0 \quad \Rightarrow \quad Nh_j + H_k s^k{}_j = 0$$

(2.8d) $$\Lambda^i{}_{\mu} \lambda^{\mu}{}_j = 0 \quad \Rightarrow \quad NV^i h_j + S^i{}_k s^k{}_j = \delta^i{}_j$$

Appealing to the duality argument provides us with the inverse relations

(2.9a) $$\lambda^0{}_{\mu} \Lambda^{\mu}{}_0 = 1 \quad \Rightarrow \quad nN + Nh_k V^k = 1$$

(2.9b) $$\lambda^i{}_{\mu} \Lambda^{\mu}{}_0 = 0 \quad \Rightarrow \quad nv^i + s^i{}_k V^k = 0$$

(2.9c) $$\lambda^0{}_{\mu} \Lambda^{\mu}{}_j = 0 \quad \Rightarrow \quad nH_j + h_k S^k{}_j = 0$$

(2.9d) $$\lambda^i{}_{\mu} \Lambda^{\mu}{}_j = 0 \quad \Rightarrow \quad nv^i H_j + s^i{}_k S^k{}_j = \delta^i{}_j$$

Some general consequences can be drawn from eqs. (2.8)-(2.9). Subtracting eq. (2.9a) from eq. (2.8a) yields

(2.10) $$nH_k v^k = Nh_k V^k$$

Finally we remark that by means of eq. (2.8b) the transformation (2.3) can be written as

(2.11a) $$X^i = S^i{}_j (x^j - v^j t)$$

(2.11b) $$T = H_k x^k + Nt$$

We remark that the form (2.11) of the frame transformation follows from the velocity addition theorem and from algebraic considerations only.

As discussed in [3] and revisited in II, further simplifications can be achieved by imposing suitable conditions on the transformation matrix $S^i{}_j$, namely the three kinematical conditions (Kin 1), (Kin 2), (Kin3) and the rotational invariance around the velocity v^i. The conclusion is that the most general frame transformation, we shall call Mansouri-Sexl transformation, is singled out by a matrix $S^i{}_j$ given by:

(2.12) $$S^i{}_j = d\delta^i{}_j + (b-d)\Pi^i{}_j = b\Pi^i{}_j + d\,\Omega^i{}_j$$

where $b(\mathbf{v})$ and $d(\mathbf{v})$ are arbitrary functions of $\mathbf{v}$ and $\Pi^i{}_j$ and $\Omega^i{}_j$ are the parallel and the orthogonal projection operators.

Before proceeding it is convenient to recall here the definitions and the properties of such projection operators discussed in I. The parallel projector $\Pi^i{}_j$ and the orthogonal projector $\Omega^i{}_j$ are given by

(2.13) $$\Pi^i{}_j = \frac{v^i v_j}{v^2}$$

(2.14) $$\Omega^i{}_j = \delta^i{}_j - \frac{v^i v_j}{v^2}$$

Of course they behave as complete set of projectors that is

(1.15a) $$\Pi^i{}_k \Pi^k{}_j = \Pi^i{}_j$$

(2.15b) $$\Omega^i{}_k \Omega^k{}_j = \Omega^i{}_j$$

(2.15c) $$\Pi^i{}_k \Omega^k{}_j = \Omega^i{}_k \Pi^k{}_j = 0$$

(2.15d) $$\Pi^i{}_j + \Omega^i{}_j = \delta^i{}_j$$

as it is easy to prove by direct calculation. Moreover for any vector a^i we set

(2.16a) $$a^i_{\parallel} = \Pi^i{}_j a^j$$

(2.16b) $$a^i_{\perp} = \Omega^i{}_j a^j$$

3. Links between the parameters of the Mansouri-Sexl transformation and its inverse

Our goal is now that of exploiting relations (2.8)-(2.9) in order to find the explicit links between the various quantities involved in the transformation (2.11) and its inverse by assuming explicitly that eq. (2.12) holds true. In the sequel, we first write down the relevant formulas followed by their proof.

(3.1) $$V^i = -\frac{b}{N}v^i\,; \qquad v^i = -\frac{N}{b}V^i$$

Using eqs. (2.8b) and (2.12) we readily get $NV^i + (b\,\Pi^i{}_j + d\,\Omega^i{}_j)v^j = 0$, namely $NV^i + b\,v^i = 0$ which implies eqs. (3.1).

(3.2) $$S^i{}_k v^k = b\,v^i\,; \qquad S^i{}_k V^k = b\,V^i$$

Both relations are a straightforward consequence of eq. (2.8b) and eq. (3.1).

(3.3) $$s^i{}_k V^k = \frac{nN}{b}\,V^i\,; \qquad s^i{}_k v^k = \frac{nN}{b}v^i$$

Both relations are a straightforward consequence of eq. (2.9b) and eq. (3.1).

(3.4a) $$h_i^{\parallel} = -\frac{n}{b}H_i^{\parallel}\,, \qquad h_i^{\perp} = -\frac{n}{d}H_i^{\perp}$$

(3.4b) $$H_i^{\parallel} = -\frac{b}{n}h_i^{\parallel}\,, \qquad H_i^{\perp} = -\frac{d}{n}h_i^{\perp}$$

Substituting eq. (2.12) into eq. (2.9c) and using eq. (2.15d) we obtain

$$n\,H_i + h_k\,(b\,\Pi^k{}_i + d\,\Omega^k{}_i) = 0 \quad \Rightarrow \quad n\,(\Pi^k{}_i + \Omega^k{}_i)H_i + h_k\,(b\,\Pi^k{}_i + d\,\Omega^k{}_i) = 0$$

In view of definitions (2.16) we can write $n\,H_i^{\parallel} + n\,H_i^{\perp} + b\,h_i^{\parallel} + d\,h_i^{\perp} = 0$ which implies eqs. (3.4).

(3.5) $$h_i = -\frac{n}{b}H_i^{\parallel} - \frac{n}{d}H_i^{\perp}$$

This equation is a straightforward consequence of eq. (3.4a).

(3.6) $$H_i = -\frac{b}{n}h_i^{\parallel} - \frac{d}{n}h_i^{\perp}$$

This equation is a straightforward consequence of eq. (3.4b).

(3.7) $$s^i{}_j = \frac{1}{d}\left(\delta^i{}_j + \frac{d-b}{b}\Pi^i{}_j\right) + v^i h_j$$

Eq. (2.9d) can be written as $s^i{}_k S^k{}_j = \delta^i{}_j - n\,v^i H_j$. Using eqs. (2.12) and (2.13) we can calculate the quantity $s^i{}_k S^k{}_j$:

$$s^i{}_k S^k{}_j = s^i{}_k[d\delta^k{}_j + (b-d)\Pi^k{}_j] = d\,s^i{}_j + \frac{b-d}{v^2}s^i{}_k v^k v_j$$

which, in view of the second eq. (3.3), namely $s^i{}_k v^k = (nN/b)v^i$, becomes

$$s^i{}_k S^k{}_j = d\,s^i{}_j + (b-d)\frac{nN}{b}\Pi^i{}_j$$

On using eq (2.9d), written above, and recalling that, in view of eq. (2.8a) , $N = 1/n - H_k v^k$, we get

$$s^i{}_j = \frac{1}{d}\left[\delta^i{}_j - n v^i H_j - (b-d)\frac{nN}{b}\Pi^i{}_j\right] = \frac{1}{d}\left[\delta^i{}_j - n v^i H_j - \frac{n(b-d)}{b}\left(\frac{1}{n} - H_k v^k\right)\Pi^i{}_j\right]$$

$$= \frac{1}{d}\left[\delta^i{}_j - n v^i H_j - \frac{b-d}{b}\Pi^i{}_j + \frac{n(b-d)}{b}H_k v^k \Pi^i{}_j\right] = \frac{1}{d}\left(\delta^i{}_j + \frac{d-b}{b}\Pi^i{}_j\right) - \frac{n}{bd}v^i\left(b H_j - (b-d)H_k\frac{v^k v_j}{v^2}\right)$$

$$= \frac{1}{d}\left(\delta^i{}_j + \frac{d-b}{b}\Pi^i{}_j\right) - \frac{n}{bd}v^i(b H^{\parallel}_j + b H^{\perp}_j - b H^{\parallel}_j + d H^{\parallel}_j) = \frac{1}{d}\left(\delta^i{}_j + \frac{d-b}{b}\Pi^i{}_j\right) - v^i\left(\frac{n}{d}H^{\perp}_j + \frac{n}{b}H^{\parallel}_j\right)$$

On account of eq. (3.5) we easily recognize that the last term within round bracket is nothing but $-h_j$. Thus eq. (3.7) is fully proved.

4. Properties of the spatial matrices

First of all, let us express the matrix $s^i{}_j$ in a form equivalent to eq. (3.7). Splitting h_i as $h^{\parallel}_i + h^{\perp}_i$ and using eq. (2.15d), we can write eq. (3.7) as follows

$$s^i{}_j = \frac{1}{d}\left(\Pi^i{}_j + \Omega^i{}_j + \frac{d}{b}\Pi^i{}_j - \Pi^i{}_j\right) + v^i h^{\parallel}_j + v^i h^{\perp}_j = \frac{1}{b}\Pi^i{}_j + \frac{1}{d}\Omega^i{}_j + (v^k h_k)\Pi^i{}_j + v^i h^{\perp}_j$$

$$= \left(\frac{1}{b} + v^k h_k\right)\Pi^i{}_j + \frac{1}{d}\Omega^i{}_j + v^i h^{\perp}_j$$

Using the second of eq. (3.1) and eq. (2.9a) we find that $(1/b) + v^k h_k = (1/b) - (N/b)V^k h_k = (1/b)(1 - N V^k h_k) = nN/b$. Therefore we arrive at the formula

$$s^i{}_j = \frac{nN}{b}\Pi^i{}_j + \frac{1}{d}\Omega^i{}_j + v^i h^{\perp}_j$$

To sum up the results obtained so far we write here the final expressions for the matrices $S^i{}_j$ and $s^i{}_j$

(4.1) $$S^i{}_j = b\Pi^i{}_j + d\,\Omega^i{}_j$$

(4.2) $$s^i{}_j = \frac{nN}{b}\Pi^i{}_j + \frac{1}{d}\Omega^i{}_j + v^i h^{\perp}_j$$

On account of the properties (2.15a, b, c) of the two projectors $\Pi^i{}_j$ and $\Omega^i{}_j$ it is quite easy to write down the inverse matrices of $S^i{}_j$ and $s^i{}_j$. Explicitly we have

(4.3) $$(S^{-1})^i{}_j = \frac{1}{b}\Pi^i{}_j + \frac{1}{d}\Omega^i{}_j$$

(4.4) $$(s^{-1})^i{}_j = \frac{b}{nN}\Pi^i{}_j + d\,\Omega^i{}_j - \frac{bd}{nN}v^i h^{\perp}_j$$

A remark is now in order. Note first that eq. (4.3) can be written as

$$(S^{-1})^i{}_j = \frac{1}{b}\Pi^i{}_j + \frac{1}{d}\Omega^i{}_j = \frac{1}{b}\Pi^i{}_j + \frac{1}{d}(\Omega^i{}_j + \Pi^i{}_j) - \frac{1}{d}\Pi^i{}_j = \frac{d-b}{bd}\Pi^i{}_j + \frac{1}{d}\delta^i{}_j = \frac{1}{d}\left(\delta^i{}_j + \frac{d-b}{b}\Pi^i{}_j\right)$$

Comparison of this relation with eq. (3.7) shows that condition $(S^{-1})^i{}_j = s^i{}_j$ holds true if and only if $h_j = 0$, namely if and only if the transformation is a Marinov transformation [4].

Finally we note that, in view of eqs (2.16), for any vector a^i we have

(4.5) $$S^i{}_j a^j = b\,a^i_{\parallel} + d\,a^i_{\perp}$$

(4.6) $$s^i{}_j a^j = \frac{nN}{b} a^i_{\parallel} + \left(\frac{1}{d}\delta^i{}_j + v^i h_j\right) a^j_{\perp}$$

Of course such relations embody formulas (3.2), (3.3).

References

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